Published by Institute of Physics Publishing for SISSA
Received: *May 31, 2005*
Revised: *October 26, 2005*
Accepted: *November 5, 2005*
Published: *November 16, 2005*

A black hole instability in five dimensions?

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Abstract: We study the moduli-space scattering of a two-charge supertube in the background of a rotating BPS D1-D5-P black hole in 4+1 dimensions, extending the static analysis of Bena and Kraus [1]. While the magnetic forces associated with this motion change the details considerably, the final conclusion is similar to that of the static analysis: we find that one can bring the supertube to the horizon, so that the BMPV black hole and the supertube merge. However, our analysis shows that this can occur even at significantly larger values of the angular momentum than was indicated by the static analysis. For a range of parameters, conservation laws and the area theorem forbid the result of the merger from being any single known object: neither near-extremal black holes nor non-supersymmetric black rings are allowed. Such results suggest that the merger triggers an instability of the rotating D1-D5-P black hole, perhaps leading to bifurcation into a pair of black objects.

Keywords: D-branes, Black Holes in String Theory.

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Contents

1. Introduction

One of the intriguing features of gravitational phenomena is the way that black holes in 3+1 dimensions differ from their higher dimensional cousins. In 3+1 dimensions, black holes are fairly simple and are associated with a variety of uniqueness and stability results. For example, one important uniqueness theorem [2] states that all stationary vacuum black hole spacetimes[1] are axisymmetric and belong to the Kerr family of solutions, parameterized by their mass M and angular momentum J. If we require that these solutions be black holes, then the mass and angular momentum are restricted by the relation $M^2 \geq |J|$. Important stability results include various proofs of perturbative stability (see, e.g., [3] and references therein) and the proof that black holes cannot bifurcate without violating a form of cosmic censorship [4]. For Schwarzschild black holes, one may also use the 2nd law of thermodynamics (and the classification of 3+1 black objects) to argue against such bifurcation since the resulting black hole fragments would necessarily have smaller total horizon area. Such an argument might plausibly forbid certain quantum mechanical process in addition to classical processes inherent in the Einstein-Hilbert dynamics.

However, the story becomes more interesting in higher dimensions. The spectrum of solutions is far richer and has proven to be full of surprises. In $d \geq 4+1$ spacetime dimensions, gravity can produce not only black holes, but also extended black objects like black strings, black branes (see, e.g. [5]) and black rings [6]. In this work, we restrict use of the term "hole" to refer to objects with spherical horizons, while 4+1 black rings

[1] I.e., those satisfying the vacuum Einstein equation.

have horizons with topology $S^1 \times S^2$. In addition, at least in certain cases such objects are known to be unstable. The classic example of an instability was discovered by Gregory and Laflamme [7, 8], who showed that black strings and branes can be dynamically unstable to a breaking of translational symmetry along the extended directions. There is also an indication [9] that rapidly rotating Myers-Perry black holes [10] in $d \geq 6$ spacetime dimensions become unstable, imposing a dynamically generated "Kerr bound" analogous to the condition $M^2 \geq |J|$ in 3+1 dimensions. Emparan and Myers [9] suggested that this instability may catalyze a transition of these black holes to a black ring or a bifurcation into a binary pair of Schwarzschild black holes (which then decays via gravitational wave emission into a single black hole of smaller angular momentum). Investigations [11 − 22] of these instabilities have given us important insights into the nature of gravity in higher dimensions. However, the general theory of instabilities remains to be fully understood.

The complicated nature of some higher dimensional solutions can make them difficult to analyze in detail. As a result, it is useful to draw inspiration from a variety of 3+1 dimensional studies, even if they do not lead immediately to rigorous results. A classic such analysis of Kerr solutions by Wald [23] probes aspects of uniqueness and stability as well as cosmic censorship. Wald considered extreme Kerr spacetimes (with $J = M^2$) to determine if one can violate the $|J| \leq M^2$ bound by, for example, dropping spinning test bodies with large spin to mass ratio into such holes (see [24] for an interesting discussion involving non-extremal holes and using charges instead of angular momentum). Note that, in the limit where an axisymmetric set of test bodies is used, it is clear that radiation of angular momentum cannot be relevant to this process.

Wald's results showed that no such violation occurs because the gravitational spin-spin repulsive force prevents such a spinning object from entering the hole. Had the opposite result been obtained, one would have been forced to conclude that throwing such an object at the black hole catalyzes a transition from a Kerr black hole to something else: either a naked singularity (violating cosmic censorship) or some new black hole solution (which does not exist).

Bena and Kraus [1] recently considered a similar gedanken experiment in 4+1 dimensions. Their analysis used a two-charge supertube [25 − 27] as a probe of the five dimensional rotating BPS black hole [31, 32] (i.e., saturating a Bogomolnyi-Prasad-Sommerfield (BPS) bound). This black hole is often referred to as the BMPV black hole after the authors of [32]. The solution has three independent charges and self-dual angular momentum, so that the magnitude of the angular momenta in the two independent planes must be equal ($|J_1| = |J_2| = J$). The magnitude of the angular momentum is also bounded by a function of the charges: $J^2 \leq N_{D1}N_{D5}N_p$, where N_{D1}, N_{D5}, N_p are quantized charges normalized to take integer values. While this constraint on angular momenta is relaxed for non-extreme black holes [33, 34], the difference $|J_1| - |J_2|$ is nevertheless bounded by a function that may be taken to parameterize the departure from extremality.

Bena and Kraus [1] considered static supertube configurations in the BMPV background and found that some such configurations exist arbitrarily close to the black hole horizon. This suggested that one can dynamically merge the supertube and black hole into a single object. Since supertubes carry unequal angular momentum in the two planes, they argued that the result of the collision should be some new supersymmetric black hole with $|J_1| \neq |J_2|$. In contrast, they also argued that supertubes with large angular momentum (relative to the scale set by the background) are prevented from reaching the horizon, so that one is protected against forming a final object which violates the constraint $J^2 \leq N_{D1} N_{D5} N_p$.

In this work we again consider the collision of a two-charge supertube with a BMPV black hole. We extend the analysis of Bena and Kraus beyond the static case to address slowly moving supertubes in a BMPV background using the moduli space approximation. Thus, we include the effects of various velocity-dependent forces (i.e., magnetic forces) as well as the small departures from the BPS limit inherent in any dynamic process. Our methods are similar to those of [35], who consider the moduli-space scattering of objects dual to supertubes in a field theoretic (non-gravitating) description.

The details of the dynamics are quite different from those which might be inferred from the static analysis. We find that magnetic forces cause *any* BPS supertube configuration (having the same symmetries assumed in [1]) to be separated from the horizon by a wall in an effective potential, so that giving this supertube any sufficiently small velocity fails to result in a merger with the black hole. However, the potential always approaches its BPS value at the horizon. Thus, additional classical forces can lift the supertube over the wall (or it may quantum mechanically tunnel through the wall), leading to merger with the black hole with any arbitrarily small energy above the BPS bound. The end result is therefore that, in agreement with Bena and Kraus [1], one can indeed bring the supertube to the horizon of the black hole so that the two objects merge. However, in contrast to [1] we find that it is *also* possible to violate even the non-extreme version of the constraint $J^2 \leq N_{D1} N_{D5} N_p$.

Due to the kinetic energy inherent in our scattering, our final object will not saturate a BPS bound. However, one can tune parameters to make the amount of non-extremality arbitrarily small. With such tuning, and in certain parameter regimes, conservation laws and the area theorem prevent the final object from being either a known non-extreme black hole [33, 34] or any small deformation of the known BPS black rings [36 – 38] (e.g., a non-BPS black ring [39]).

However, the literature does now contain BPS solutions with the correct conserved quantities and having a larger area than the original BMPV black hole. These solutions are the BPS *pairs* of black rings studied in [40, 41]. Thus, some non-extreme deformation of these solutions could represent the final result of our collision. Here we note the sharp contrast with the 3+1 dimensional case in which splitting a black hole into fragments always reduces the total entropy; in the context of [40, 41] the break-up of BMPV into two or more black rings is entropically favored, and our collision could well trigger an instability of BMPV leading to such a fragmentation. The picture is similar to one of the scenarios discussed by Emparan and Myers in [9], except that our process would occur in 4+1 dimensions and we have studied the regime close to the BPS limit (as opposed to the uncharged regime). Note that the solutions of Gauntlett and Gutowski [40, 41] were unknown at the time of [1], and even the existence of supersymmetric black rings was in question.

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Our discussion below is organized as follows. We begin with a review in section 2 of relevant properties of BMPV black holes and some other black objects. Then, in section 3, we set up the formalism for a supertube moving in the (type IIA dual of the) BMPV background and compute the associated action in the moduli space approximation. Some details are relegated to appendix A. In section 4 we study the detailed dynamics of the supertube in the moduli space (i.e., slow-velocity) approximation. Our study considers configurations preserving a certain $U(1)$ symmetry of the background, so that this symmetry is preserved by the evolution. We then use conservation laws to reduce the problem to that of motion in an effective potential, which is straightforward to analyze. We find in this approximation that a supertube with arbitrary charges *can* merge with the BMPV black hole. We conclude in section 5 with some discussion and further consideration of the possible end results of our collision.

2. BMPV black holes and other black objects

In this section we review certain properties of BMPV black holes [31, 32] (see also [42] for a useful representation) and other relevant black objects. A BMPV black hole is a supersymmetric, rotating, asymptotically flat solution of the T^5 reduction of Type IIB supergravity to five dimensions. The lift of the BMPV solution to ten dimensions is given in the string frame by

$$\begin{aligned} ds^2 &= H_{D1}^{-1/2} H_{D5}^{-1/2} \left[-dt^2 + dz^2 + (H_p - 1)(dt - dz)^2 + 2(\gamma_1 d\phi_1 + \gamma_2 d\phi_2)(dz - dt) \right] + \\ &\quad + H_{D1}^{1/2} H_{D5}^{1/2} (ds^2_{\mathbb{R}^4}) + H_{D1}^{1/2} H_{D5}^{-1/2} (ds^2_{T^4}) \,, \qquad (2.1) \\ C_2 &= (H_{D1}^{-1} - 1) dt \wedge dz - r_{D5}^2 \cos^2\theta d\phi_1 \wedge d\phi_2 + H_{D1}^{-1}(dt - dz) \wedge (\gamma_1 d\phi_1 + \gamma_2 d\phi_2) \qquad (2.2) \\ e^{2\Phi} &= \frac{H_{D1}}{H_{D5}} \,, \qquad (2.3) \end{aligned}$$

where

$$\gamma_1 = \frac{\omega}{r^2} \sin^2\theta \qquad \text{and} \qquad \gamma_2 = \frac{\omega}{r^2} \cos^2\theta \,, \qquad (2.4)$$

and where the fields are normalized as in [43]. The solution is specified by flat metrics ($ds^2_{\mathbb{R}^4}$ and $ds^2_{T^4}$) on $\mathbb{R}^4$ and T^4, three Harmonic functions (H_{D1}, H_{D5} and H_p) and *only one* angular momentum parameter ω. Although the solution has angular momenta (J_1, J_2) in two independent planes (associated with the commuting Killing vectors $\partial_{\phi_1}, \partial_{\phi_2}$), the solution satisfies $J_1 = J_2 = \frac{\pi}{4G_5}\omega$. The internal euclidean space, T^4, is parameterized by cartesian coordinates x_6, x_7, x_8 and x_9 while the external euclidean space, $\mathbb{R}^4$, has coordinates $r, \theta, \phi_1, \phi_2$ with metric

$$ds^2_{\mathbb{R}^4} = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi_1^2 + r^2 \cos^2\theta d\phi_2^2 \,. \qquad (2.5)$$

Here r is the radial coordinate and θ, ϕ_1 and ϕ_2 are angles on S^3. These coordinates $(r, \theta, \phi_1, \phi_2)$ can be obtained from cartesian coordinates x_1, x_2, x_3, x_4 on $\mathbb{R}^4$ through:

$$x_1 + ix_2 = r \sin\theta e^{i\phi_1} \,; \qquad x_3 + ix_4 = r \cos\theta e^{i\phi_2} \,,$$

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where θ ranges from 0 to $\pi/2$ and ϕ_1, ϕ_2 ranges over $[0, 2\pi)$. The three Harmonic functions (H_{D1}, H_{D5} and H_p) on the external euclidean space $\mathbb{R}^4$ are given by

$$H_{D1} = 1 + \frac{r_{D1}^2}{r^2}\,; \qquad H_{D5} = 1 + \frac{r_{D5}^2}{r^2}\,; \qquad H_p = 1 + \frac{r_p^2}{r^2}\,, \tag{2.6}$$

and the coordinate singularity at $r = 0$ is a smooth horizon of the black hole.

Compactifying the z-direction on S^1 yields a five dimensional black hole solution with three distinct charges. Denoting the asymptotic length of the S^1 by $2\pi R_B$ and the volume of the T^4 by $(2\pi\ell)^4$, the quantized integral charges are [44] :

$$N_1 = \frac{\ell^4}{g_B \alpha'^3} r_{D1}^2, \qquad N_5 = \frac{r_{D5}^2}{g_B \alpha'}, \qquad N_P = \frac{R_B^2 \ell^4}{g_B^2 \alpha'^4} r_p^2\,. \tag{2.7}$$

where g_B is the string coupling. Here the B subscripts refer to the fact that (2.1)–(2.3) is a type IIB solution and foreshadow the fact that we will shortly dualize this solution to a IIA frame.

The ADM mass, angular momenta and the entropy of the corresponding five dimensional solution are

$$M_{BMPV} = \frac{\pi}{4G_5}\left(r_{D1}^2 + r_{D5}^2 + r_p^2\right) = \frac{1}{g_B^2}\left(\frac{R_B g_B N_{D1}}{\alpha'} + \frac{R_B \ell^4 g_B N_{D5}}{\alpha'^3} + \frac{g_B^2 N_p}{R_B}\right), \tag{2.8}$$

$$J_{\phi_1} \equiv J_1 = J = \frac{\pi}{4G_5}\omega\,, \qquad J_{\phi_2} \equiv J_2 = J = \frac{\pi}{4G_5}\omega\,, \tag{2.9}$$

$$S_{BMPV} = \frac{2\pi^2}{4G_5}\sqrt{r_{D1}^2 r_{D5}^2 r_p^2 - \omega^2} = 2\pi\sqrt{N_{D1}N_{D5}N_p - J^2}\,. \tag{2.10}$$

Here the five dimensional Newton's constant G_5 is related to ten dimensional Newton's constant $G_{10} = 8\pi^6\alpha'^4 g_B^2$ by

$$G_5 = \frac{G_{10}}{2\pi R(2\pi\ell)^4} = \frac{\pi\alpha'^4 g_B^2}{4\ell^4 R_B}\,.$$

The solution obeys a Kerr like bound

$$\omega^2 \leq r_{D1}^2 r_{D5}^2 r_p^2 \qquad \text{or, equivalently,} \qquad J^2 \leq N_{D1}N_{D5}N_p\,. \tag{2.11}$$

Note that we have set $\hbar = 1$ so that the angular momentum J takes half-integer values. When the bound (2.11) is violated, the horizon disappears to expose naked closed time-like curves.

The solution (2.1)–(2.3) will be of central use in section 3 below, where we study the moduli space scattering of supertubes in this background. However, we will also be interested in a few basic features of other solutions, such as the non-BPS version of (2.1)–(2.3) found by Cvetic and Youm [33, 34]. These non-BPS solutions are parameterized by three charges and also by three additional parameters m, L_1, and L_2 which are related respectively to the energy above extremality and to the two independent angular momenta. We will be most interested in the near-BPS limit, also studied in [33, 34]. In this limit $m \ll r_{D1}^2, r_{D5}^2, r_p^2$ and the ADM mass M_{CY} for the Cvetic-Youm solutions is given [33, 34]

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to leading non-trivial order in m by

$$M_{CY} \approx M_{BMPV} + \frac{\pi}{4G_5}\frac{m^2}{2}\left(\frac{1}{r_{D1}^2} + \frac{1}{r_{D5}^2} + \frac{1}{r_p^2}\right) + \mathcal{O}(m^2)\,. \tag{2.12}$$

The (half-integer) angular momenta are determined by the relations[2]

$$\frac{J_1 + J_2}{2} = \frac{\pi}{4G_5\sqrt{2}} r_{D1} r_{D5} r_p (L_1 + L_2) + \mathcal{O}(m^2) \tag{2.13}$$

and

$$\frac{J_1 - J_2}{2} = \frac{\pi}{4G_5\sqrt{2}} r_{D1} r_{D5} r_p \left(\frac{1}{r_{D1}^2} + \frac{1}{r_{D5}^2} + \frac{1}{r_p^2}\right)(L_1 - L_2) + \mathcal{O}(m^2)\,, \tag{2.14}$$

while the integer charges N_{D1}, N_{D5}, N_p are again given by (2.7) and, to leading non-trivial order in m, the entropy is given by

$$S_{CY} \approx S_{BMPV} + \frac{m\pi^2}{4G_5}\frac{r_{D1}^2 r_{D5}^2 + r_{D5}^2 r_p^2 + r_{D1}^2 r_p^2}{r_{D1} r_{D5} r_p}\sqrt{1 - \frac{1}{2}(L_1 - L_2)^2} + \mathcal{O}(m^2)\,, \tag{2.15}$$

where we have corrected a minor error in the expansion given in [33, 34]. It is important to note that, although these solutions carry two independent angular momenta in orthogonal planes, their difference is bounded and satisfies

$$|J_1 - J_2| \leq \frac{\pi m}{2G_5} r_{D1} r_{D5} r_p \left(\frac{1}{r_{D1}^2} + \frac{1}{r_{D5}^2} + \frac{1}{r_p^2}\right) + \mathcal{O}(m^2)\,. \tag{2.16}$$

In addition to the above black holes, a new family of supersymmetric black solutions with the same asymptotic charges and horizon topology $S^1 \times S^2$ has recently been constructed in [36 – 38]. These "black ring" solutions are labeled by 7 independent parameters. They carry three charges, N_{D1}, N_{D5} and N_p (related to 3 charge radii r_{D1}, r_{D5}, r_p), two independent angular momenta J_1 and J_2 and three dipole charges n_{D1}, n_{D5} and n_{KK}. The dipole charges are not conserved. The ADM mass and angular momenta take the following form

$$\begin{aligned} M_{\rm ring} &= \frac{\pi}{4G_5}\left(r_{D1}^2 + r_{D5}^2 + r_p^2\right) = \frac{1}{g_B^2}\left(\frac{R_B g_B N_{D1}}{\alpha'} + \frac{R_B \ell^4 g_B N_{D5}}{\alpha'^3} + \frac{g_B^2 N_p}{R_B}\right), && (2.17)\\ J_2 &= \frac{1}{2}\left(n_{D1}N_{D5} + n_{D5}N_{D1} + n_{KK}N_P - n_{D1}n_{D5}n_{KK}\right), && (2.18)\\ J_1 &= J_2 + \frac{\pi R^2}{4G_5}\left(\frac{g_B\alpha'^3}{R_B V}n_{D1} + \frac{g_B\alpha'}{R_B}n_{D5} + R_B n_{KK}\right), && (2.19) \end{aligned}$$

where R is a parameter appearing explicitly in the solution and is called the "radius" of the ring. Note that, as we will later be interested in rings for which the larger angular momentum lies in the ϕ_1 plane, we have permuted the angular momenta relative to the

[2]Note that we have chosen a convention so that J_1 and J_2 have no relative minus sign in the BPS limit. This differs from the conventions of [33, 34].

conventions of [36 – 38]. The entropy of the black ring is given by

$$S_{\text{ring}} = 2\pi\sqrt{N_{D1}N_{D5}N_p - \mathcal{N}_{D1}\mathcal{N}_{D5}\mathcal{N}_p - J_2^2 - n_{D1}n_{D5}n_{KK}(J_1 - J_2)}\,, \tag{2.20}$$

where we have used the notation $\mathcal{N}_{D1} := N_{D1} - n_{D1}n_{KK}$, $\mathcal{N}_{D5} := N_{D5} - n_{D5}n_{KK}$ and $\mathcal{N}_p := N_p - n_{D1}n_{D5}$. We have normalized both the monopole charges N_{D1}, N_{D5}, N_p and the dipole charges n_{D1}, n_{D5}, n_{KK} so that they take integer values and, as usual, the angular momenta J_1, J_2 take half-integer values.

In the limit $R \to 0$ the black ring solution reduces to the BMPV solution. However, the horizon area is discontinuous at $R = 0$, where the horizon changes topology from the $S^1 \times S^2$ of the black ring to the S^3 of the BMPV black hole. An analysis of (2.20) shows [38] that the $R \to 0$ limit of the black ring horizon area is always less than that of the corresponding BMPV black hole. One may also show by further calculation that, for fixed asymptotic charges N_{D1}, N_{D5}, N_p, the entropy of the black ring is maximized for

$$\begin{aligned} n_{D1} &= \sqrt{\frac{N_{D1}N_p}{3N_{D5}}}\,; \qquad n_{D5} = \sqrt{\frac{N_{D5}N_p}{3N_{D1}}}\,; \\ n_{KK} &= \sqrt{\frac{N_{D1}N_{D5}}{3N_p}} \quad \text{and} \quad J_1 = J_2 = \frac{4\sqrt{N_{D1}N_{D5}N_p}}{3\sqrt{3}}\,. \end{aligned} \tag{2.21}$$

This maximal entropy is given by

$$S_{\text{ring}}(\max) = 2\pi\frac{\sqrt{N_{D1}N_{D5}N_p}}{3}\,. \tag{2.22}$$

Since the two angular momenta (2.21) are equal, the configuration is not really a black ring, but the result (2.22) does provide an upper bound on the entropy of all black rings with fixed N_{D1}, N_{D5}, N_p. Note that (2.22) is also the entropy of a BMPV black hole with a larger angular momentum ($J = \frac{2\sqrt{2}}{3}\sqrt{N_{D1}N_{D5}N_p}$).

A particularly interesting feature of BPS black rings is that one can sometimes increase their entropy [40] by splitting them into several BPS black rings with smaller values of the quantized charges N_{D1}, N_{D5}, N_p. In particular, the results of Gauntlett and Gutowski [40] indicate that for any BMPV black hole with $J > 0$, there are pairs of black rings whose asymptotic charges and angular momenta sum to the same values as those of the black hole and for which the sum of the black ring horizon areas is *larger* than that of the BMPV black hole. The upper bound on the entropy of a single black ring mentioned above thus means that one can also increase the entropy of black rings with small radius R by replacing them with a pair of black rings having the same total asymptotic charges and angular momenta. Thus, at least at small radius R one might expect black rings to be unstable to fragmentation.

3. Supertubes and the action on moduli space

In this section we review certain properties of two-charge supertubes (section 3.1) and obtain the moduli space action (section 3.2) for a slowly moving supertube in the BMPV background.

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3.1 Two-charge supertubes

As with all stringy objects, supertubes admit a variety of dual descriptions. We may therefore choose any duality frame which is convenient for the purpose at hand. Our goal below is to describe the supertube dynamics in a simple and familiar way. We therefore choose the IIA duality frame in which the tube carries D0 and F1 charges, as in this frame the tube admits a description in terms of the D2-brane Born-Infeld action. Such a description was used in the original work [25 – 27], in which the supertubes are described as cylindrical D2 branes supported against collapse by the angular momentum associated with crossed electric and magnetic fields (which account for the F1 and D0 charges). See also [28 – 30].

We remind the reader that, at each time, such tubes have topology $S^1_{\text{cross}} \times \mathbb{R}$ or, if they wrap a compact S^1, they have topology $S^1_{\text{cross}} \times S^1_z$. Here we have distinguished the factor S^1_{cross} (the "cross-section" of the tube) from the factor S^1_z which is always associated with a translational symmetry. Below, this symmetry corresponds to shifts of the coordinate z. The factor S^1_z is also the direction along which the F1 strings are extended. When propagating in Minkowski space, the cross section of a supertube can be an arbitrary (non-intersecting) closed curve; all such configurations saturate a BPS bound associated with their F1 and D0 charge. Due to their topology, such supertubes carry no net D2-brane charge, but they do carry a D2 dipole moment.

The Born-Infeld action for a D2 brane is given by

$$\begin{aligned} S &= S_{DBI} + S_{WZ} = \int \mathcal{L} d^3x \\ &= -\tau_{D2} \int d^3x e^{-\Phi} \sqrt{-\det(g_{ab} + b_{ab} + F_{ab})} + \tau_{D2} \int c \wedge e^{F_2 + b_2} , \end{aligned} \tag{3.1}$$

where g_{ab} and b_{ab} are respectively the metric and the Neveu-Schwarz two-form field induced on the supertube by immersion in some background. Similarly, c_1, c_3 etc. are the fields induced on the supertube by the background Ramond-Ramond fields and Φ is the background dilaton. Finally, $F = \frac{1}{2} F_{ab} dx^a \wedge dx^b$ is the D2-brane world-volume gauge-field and τ_{D2} is the D2-brane tension $\tau_{D2} = \frac{1}{(2\pi)^2 g_A \alpha'^{3/2}}$, where g_A is the string coupling in our present type IIA duality frame.

Due to the gauge field F_{ab}, the D2-brane will carry D0-brane charge and F1-string charges:

$$q_{D0} = \frac{\tau_{D2}}{\tau_{D0}} \int dz \int d\sigma F_{\sigma z} \quad \text{and} \quad q_{F1} = \frac{1}{\tau_{F1}} \int d\sigma \frac{\partial \mathcal{L}}{\partial F_{tz}} , \tag{3.2}$$

where $\tau_{D0} = \frac{1}{g_A \sqrt{\alpha'}}$ and $\tau_{F1} = \frac{1}{2\pi\alpha'}$ are the D0 and F1 brane tensions respectively. We have normalized q_{F1} and q_{D0} so that they take integer values.

Supertubes are supersymmetric configurations, which we may take to be time-independent. Let us briefly review the properties of the original supertubes [25] associated with embedding the D2-brane in Minkowski space. Such configurations have gauge fields of the form [25]

$$F = F_{tz} dt \wedge dz + F_{\sigma z} d\sigma \wedge dz , \tag{3.3}$$

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where σ is a coordinate on the S^1 cross-section and z is a coordinate along the $\mathbb{R}$ direction. In particular, supersymmetry requires $F_{tz} = 1$ in string units (with $2\pi\alpha' = 1$). However, the magnetic field $F_{\sigma z}$ can be an arbitrary non-vanishing function of σ. We take $F_{\sigma z}$ to be positive below. Under such conditions, the energy constructed from (3.1) saturates a BPS bound:

$$E = \tau_{D0}|q_{D0}| + \tau_{F1}|q_{F1}|2\pi R_A\,, \tag{3.4}$$

where $2\pi R_A$ is the length S^1_z. This result shows that the positive energies associated with the D2-brane tension and with the rotation are exactly canceled by the negative binding energy of D0 and F1 branes and suggests that supertubes may be interpreted as marginal bound states[3] of D0 branes and F1 strings. If one also imposes rotational symmetry around the S^1, then a singly wound such supertube has angular momentum $j = q_{F1}q_{D0}$ in the plane of the S^1.

3.2 Moduli space action

Our goal is to study the moduli-space motion of a supertube in the BMPV black hole background. However, we have presented the supertube above in a IIA duality frame while the BMPV background (2.1)–(2.3) was given in a IIB frame. As a result, it is convenient to T-dualize the BMPV background (2.1)–(2.3) along the D1-branes following [47–49]. The result is:

$$\begin{aligned} ds^2 \;=&\; -H_{D0}^{-1/2}H_{D4}^{-1/2}H_{F1}^{-1}\,(dt+\gamma_1 d\phi_1+\gamma_2 d\phi_2)^2 + H_{D0}^{1/2}H_{D4}^{1/2}H_{F1}^{-1}dz^2 + \\ &+ H_{D0}^{1/2}H_{D4}^{1/2}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi_1^2+\cos^2\theta d\phi_2^2) + H_{D0}^{1/2}H_{D4}^{-1/2}ds^2_{T_4} \end{aligned} \tag{3.5}$$

while the dilaton and the gauge fields are

$$e^{2\Phi} = \frac{H_{D0}^{3/2}}{H_{D4}^{1/2}H_{F1}} \tag{3.6}$$

$$C_1 = (H_{D0}^{-1}-1)dt + H_{D0}^{-1}(\gamma_1 d\phi_1+\gamma_2 d\phi_2) \tag{3.7}$$

$$C_3 = -(H_{D4}-1)r^2\cos^2\theta d\phi_1\wedge d\phi_2\wedge dz + H_{F1}^{-1}dt\wedge(\gamma_1 d\phi_1+\gamma_2 d\phi_2)\wedge dz \tag{3.8}$$

$$B_2 = (H_{F1}^{-1}-1)dt\wedge dz + H_{F1}^{-1}(\gamma_1 d\phi_1+\gamma_2 d\phi_2)\wedge dz\,. \tag{3.9}$$

In (3.5)–(3.9), we have introduced $H_{D0} = H_{D1}, H_{D4} = H_5, H_{F1} = H_p$ with $r_{D0} = r_{D1}, r_{D4} = r_{D5}, r_{F1} = r_p$ in order to reflect the charge structure in the IIA frame. The quantized charges in this duality frame are obtained by applying T-duality to the expressions (2.7):

$$N_{D0} = N_1 = \frac{\ell^4 R_A}{g_A\alpha'^{7/2}}r_{D0}^2\,, \quad N_{D4} = N_{D5} = \frac{R_A}{g_A\alpha'^{3/2}}r_{D4}^2\,, \quad N_{F1} = N_p = \frac{\ell^4}{g_A^2\alpha'^3}r_{F1}^2\,, \tag{3.10}$$

where $g_A = g_B\frac{\sqrt{\alpha'}}{R_B}$ is the Type IIA string coupling and $R_A = \frac{\alpha'}{R_B}$.

[3] See [45] for a demonstration in this duality frame that the BPS states described above are in fact bound; i.e., that the spectrum of states is discrete despite the arbitrary shape of $S^1_{\rm cross}$. See also [46].

Let us now consider test D2-branes in the background (3.5–3.9). So long as the z directions on the world-volume and in the background coincide, one finds [1] that static supersymmetric tubes have properties similar to the those in Minkowski space described in section 3.1. In particular, if the worldvolume gauge fields again take the form (3.3), with $F_{tz} = 1$, the configuration is supersymmetric for any $F_{\sigma z}$ and any embedding of the S^1 into the external spatial directions $r, \theta, \phi_1, \phi_2$ at a fixed value of the internal torus coordinates (x_6, x_7, x_8, x_9). The energy of such tubes is again given by the BPS bound (3.4).

Now, we wish to use a supertube to dynamically probe the IIA BMPV background (3.5)–(3.9). The motion of such tubes will break supersymmetry, so that we will be forced to generalize the static ansatz above. However, we may simplify the analysis by imposing translational symmetry in both σ and z. Thus, the world-volume gauge field takes the form

$$F = F_{tz} dt \wedge dz + F_{\sigma z} d\sigma \wedge dz + F_{t\sigma} dt \wedge d\sigma \,, \tag{3.11}$$

where F_{tz}, $F_{t\sigma}$, and $F_{\sigma z}$ are functions of time alone. We will work in static gauge; i.e. we take t and z to coincide on the worldvolume and in the spacetime. The supertube is then described by the ansatz

$$\begin{aligned} t_{\text{spacetime}} &= t_{\text{supertube}} \\ z_{\text{spacetime}} &= z_{\text{supertube}} \,. \end{aligned} \tag{3.12}$$

We will also take the supertube to remain at an arbitrary (fixed) location on the T^4. The rest of the coordinates $(r, \theta, \phi_1, \phi_2)$ are functions of σ and t alone.

The action governing the dynamics of the slowly moving supertube is then obtained by inserting the ansatz (3.11)–(3.12) and the IIA BMPV background (3.5)–(3.9) into the D2-brane action (3.1). The dynamical variables are the coordinates of the supertube and the deviations of the components of the 2-form field F_2 from their supersymmetric values. In particular, we denote the deviation of the electric field F_{tz} by $\delta F_{tz} := F_{tz} - 1$.

We now carry out an expansion to second order in the velocities and in the quantities δF_{tz} and $F_{t\sigma}$. For the Dirac-Born-Infeld part of the action (the first term on the last line of (3.1)), the calculation is somewhat involved and is outlined in the appendix. However, the remaining (Wess-Zumino) term is more straightforward and yields:

$$\begin{aligned} S_{WZ} &= \tau_{D2} \int c_3 + \tau_{D2} \int c_1 \wedge (F+b)_2 \\ &= \tau_{D2} \int c_{3t\sigma z} + \tau_{D2} \int c_{1t} (F+b)_{2\sigma z} - \tau_{D2} \int c_{1\sigma} (F+b)_{2tz} \\ &= \tau_{D2} \int dt d\sigma dz \Big[-\delta F_{tz} H_{D0}^{-1} \gamma_\sigma + (H_{D0}^{-1} - 1 + \gamma_t H_{D0}^{-1}) F_{\sigma z} - (H_{D4} - 1) r^2 \cos^2\theta \, \times \\ &\qquad \times \left(\frac{\partial \phi_1}{\partial t} \frac{\partial \phi_2}{\partial \sigma} - \frac{\partial \phi_1}{\partial \sigma} \frac{\partial \phi_2}{\partial t} \right) \Big] \,, \end{aligned} \tag{3.13}$$

where

$$\gamma_\xi := \gamma_1 \frac{\partial \phi_1}{\partial \xi} + \gamma_2 \frac{\partial \phi_2}{\partial \xi} \,. \tag{3.14}$$

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Note that although (3.13) is the full Wess-Zumino part of the action, it is linear in both the velocities and δF_{tz} and it is independent of $F_{t\sigma}$. Adding the Wess-Zumino term (3.13) to the DBI term (A.8) we obtain the complete action (A.9) to quadratic order. The result is:

$$\begin{aligned} S = \tau_{D2} \int dtdzd\sigma \Bigg[& -F_{\sigma z} - H_{D4}\Delta_{\sigma t} + \left\{ \frac{H_{D4}\Delta_{\sigma\sigma}}{F_{\sigma z}} \right\} \delta F_{tz} + \frac{H_{D4}F_{t\sigma}^2}{2F_{\sigma z}} - (H_{D4}-1)r^2\cos^2\theta \times \\ & \times \left(\frac{\partial\phi_1}{\partial t}\frac{\partial\phi_2}{\partial\sigma} - \frac{\partial\phi_1}{\partial\sigma}\frac{\partial\phi_2}{\partial t} \right) + \frac{H_{D4}H_{F1}\Sigma}{2F_{\sigma z}} \times \\ & \times \left(\Delta_{tt} - \frac{2\Delta_{\sigma t}\delta F_{tz}}{F_{\sigma z}} + \frac{\Delta_{\sigma\sigma}\delta F_{tz}^2}{F_{\sigma z}^2} \right) \Bigg], \end{aligned} \tag{3.15}$$

where we have introduced

$$\begin{aligned} \Sigma &:= F_{\sigma z}^2 + 2\gamma_\sigma F_{\sigma z} H_{F1}^{-1} + H_{F1}^{-1} H_{D0} H_{D4} \Delta_{\sigma\sigma} \qquad \text{and} \\ \Delta_{\xi\eta} &:= \frac{\partial r}{\partial \xi}\frac{\partial r}{\partial \eta} + r^2 \frac{\partial\theta}{\partial\xi}\frac{\partial\theta}{\partial\eta} + r^2\sin^2\theta \frac{\partial\phi_1}{\partial\xi}\frac{\partial\phi_1}{\partial\eta} + r^2\cos^2\theta\frac{\partial\phi_2}{\partial\xi}\frac{\partial\phi_2}{\partial\eta}. \end{aligned} \tag{3.16}$$

The action (3.15) is the key result for the discussion below. It is valid for an arbitrary circular embedding centered on the BMPV black hole. Setting velocities to zero and values of the worldvolume fields to their supersymmetric values, the action (3.15) reduces to the result of [1]:

$$S_{\text{susy}} = -\tau_{D2} \int dtdzd\sigma F_{\sigma z} = -\tau_0 \int q_{D0} dt\,. \tag{3.17}$$

A quantity of great interest is the F1 charge, obtained by differentiating the lagrangian (3.15) with respect to F_{tz} (see 3.2). We find

$$q_{F1} = \frac{\tau_{D2}}{\tau_{F1}} \int d\sigma \left[\frac{H_{D4}\Delta_{\sigma\sigma}}{F_{\sigma z}} - \frac{H_{D4}H_{F1}\Delta_{\sigma t}\Sigma}{F_{\sigma z}^2} + \frac{\Delta_{\sigma\sigma}H_{D4}H_{F1}\Sigma}{F_{\sigma z}^3}\delta F_{tz} \right]. \tag{3.18}$$

Setting $\delta F_{tz} = 0$ and velocities to zero leaves just the first term in (3.18), which again reproduces the results of [1].

The dynamics of the world-volume field F must ensure conservation of the various supertube charges. To see this for the D0 charge, recall that we have taken F independent of both σ and z, i.e. $\partial_z F_{ab} = \partial_\sigma F_{ab} = 0$. The Bianchi identity

$$\partial_t F_{\sigma z} + \partial_z F_{t\phi} + \partial_\sigma F_{zt} = 0 \tag{3.19}$$

then implies

$$\partial_t F_{\sigma z} = 0\,. \tag{3.20}$$

Integrating this result over the D2-brane worldvolume yields conservation of the D0 charge.

The other conservation laws follow from the equations of motion:

$$0 = \partial_a \frac{\partial\mathcal{L}}{\partial F_{ab}} = \partial_t \frac{\partial\mathcal{L}}{\partial F_{tb}}, \tag{3.21}$$

where in the last step we have used the assumed translation invariance in σ and z. Integrating (3.21) over an appropriate cross section of the D2 worldvolume yields conservation of an F1 charge in the b-direction on the D2-brane. For $b = z$, it yields $\frac{dq_{F1}}{dt} = 0$, while for

$b = \sigma$ we find

$$\frac{\partial \mathcal{L}}{\partial F_{t\sigma}} = \tau_{D2} \frac{H_{D4} F_{t\sigma}}{F_{\sigma z}} = \text{const}\,. \tag{3.22}$$

From (3.22) we see that, due to our assumed translation invariance, it is consistent to also set $F_{t\sigma}$ to zero.

4. Scattering

In this section we use the action (3.15) to understand the dynamics of the supertube as it moves in the black hole background. Recall that we are interested in the dependence of the result on the angular momentum of the supertube. In particular, we are interested in whether the dynamics allows the supertube to merge with the black hole when the total angular momentum does not correspond to that of another BMPV black hole. Note that we have assumed sufficient symmetries to reduce the problem to five spacetime dimensions, where angular momenta are classified by two-independent parameters associated with the block diagonalization of the angular momentum two-form. It is simplest to investigate this question in the case where the angular momenta of the BMPV black hole and the supertube are simultaneously block diagonalizable. Since the angular momentum of a static circular supertube always lies in some plane, this occurs only when the plane of the supertube coincides with one of the principal angular momentum planes of the black hole. Since the black hole is symmetric under exchange of these planes, we may as well follow [1] and study embeddings with r, θ, ϕ_2 independent of σ. We fix the remaining reparametrization freedom by requiring

$$\phi_1(\sigma, t) = \sigma\,. \tag{4.1}$$

Such embeddings provide the σ-translation symmetry assumed above, which in turn guarantees that this ansatz is consistent. We also impose $F_{t\sigma} = 0$, which is consistent by (3.22). Since our ansatz is determined entirely by (compact) symmetries of the background, the dynamics for our ansatz follows from the action (3.15) restricted to such embeddings. The result is (where dot denotes the derivatives with respect to time)

$$\begin{aligned} S = (2\pi)^2 R_A \tau_{D2} \int dt \Bigg[& -F_{\sigma z} + r_{D4}^2 \cos^2\theta \dot{\phi}_2 + \delta F_{tz} \frac{H_{D4} r^2 \sin^2\theta}{F_{\sigma z}} + \\ & + \left\{ \frac{F_{\sigma z} H_{D4} H_{F1}}{2} + H_{D4}\gamma_1 + \frac{H_{D0} H_{D4}^2 r^2 \sin^2\theta}{2F_{\sigma z}} \right\} \times \\ & \times \left(\frac{\delta F_{tz}^2 r^2 \sin^2\theta}{F_{\sigma z}^2} + \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \cos^2\theta \dot{\phi}_2{}^2 \right) \Bigg]\,. \end{aligned} \tag{4.2}$$

Similarly, the F1 charge of our configuration is

$$q_{F1} = 2\pi \frac{\tau_{D2}}{\tau_{F1}} \frac{H_{D4} r^2 \sin^2\theta}{F_{\sigma z}} \left[1 + 2 \left\{ \frac{F_{\sigma z} H_{F1}}{2} + \gamma_1 + \frac{H_{D0} H_{D4} r^2 \sin^2\theta}{2F_{\sigma z}} \right\} \frac{\delta F_{tz}}{F_{\sigma z}} \right]. \tag{4.3}$$

Finally, from Noether's theorem and our ansatz above we find the angular momenta

$$j_1 = \int dz d\sigma \frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = q_{D0} q_{F1}\,, \qquad \text{and} \tag{4.4}$$

$$j_2 = \int dz d\sigma \frac{\partial \mathcal{L}}{\partial \dot{\phi}_2} = (2\pi)^2 R_A \tau_{D2} \Bigg[r_{D4}^2 \cos^2\theta + \frac{H_{D4}\cos^2\theta \dot{\phi}_2}{F_{\sigma z}} \times \left(F_{\sigma z}^2 H_{F1} r^2 + 2F_{\sigma z}\omega \sin^2\theta + H_{D0}H_{D4}r^4 \sin^2\theta \right) \Bigg], \tag{4.5}$$

associated with ϕ_1 and ϕ_2.

Now, the dynamical variables in the action (4.2) are $r, \theta, \phi_2, F_{\phi z}$ and F_{tz} (we have already set $F_{t\sigma} = 0$). Recall that the conservation laws for the D0 and F1 charges provide two integrals of the motion. Further conservation laws follow from the three Killing vectors $\partial_{\phi_1}, \partial_{\phi_2}$ and ∂_t of the background (3.5), though conservation of the angular momentum associated with ∂_{ϕ_1} turns out to be trivial for our ansatz. Thus we have 4 (useful) conserved quantities and 5 dynamical variables.

Before analyzing the resulting dynamics, it is useful to note that the surface $\theta = \pi/2$ is fixed under translations of ϕ_2. Thus, any departure from $\theta = \pi/2$ would entail a breaking of this symmetry and we may consistently restrict the dynamics to $\theta = \pi/2$. In this case, we still have 4 useful conserved quantities, but we have only 4 dynamical variables. Thus, the motion of the supertube reduces to quadratures. We explore this simple special case in section 4.1 below before moving on to the general case in section 4.2

4.1 A special case: collision in the $\theta = \pi/2$ plane

In this subsection we analyze the motion of the supertube when restricted to move only in the $\theta = \pi/2$ plane. Note that ϕ_2 drops out of the analysis entirely as choosing $\theta = \pi/2$ forces the supertube to lie on the 'axis' where ϕ_2 is ill-defined. It is useful to define the quantity $\mu_{F1} := q_{F1}\tau_{F1}/2\pi\tau_{D2}$, proportional to the F1 charge. One may then solve (4.3) for δF_{tz} as a function of r and other quantities:

$$\delta F_{tz} = F_{tz} - 1 = \frac{(\mu_{F1}F_{\sigma z} - H_{D4}r^2)F_{\sigma z}^2}{H_{D4}(F_{\sigma z}^2 H_{F1} r^2 + 2F_{\sigma z}\omega + H_{D0}H_{D4}r^4)}\,. \tag{4.6}$$

When $F_{tz} = 1$, the right hand side of (4.6) must vanish. This happens precisely at the point where one satisfies $F_{\sigma z}\mu_{F1} = H_{D4}r^2$, which is identical to the relation between r and μ_{F1} that would be imposed by supersymmetry. Thus, $\delta F_{tz} = 0$ at the minimum of the effective potential for fixed μ_{F1}.

Using (4.6) and $\theta = \pi/2$, the energy constructed from (4.2) is $E = \tau_{D0}|q_{D0}| + 2\pi R_A \tau_{F1}|q_{F1}| + \Delta E$, where ΔE is the energy above the BPS limit:

$$\Delta E = (2\pi)^2 \tau_{D2} R_A \left[\frac{F_{\sigma z} r^2 (F_{\sigma z}\mu_{F1} - H_{D4}r^2)^2}{2H_{D4}r^2(2F_{\sigma z}\omega + F_{\sigma z}^2 H_{F1} r^2 + H_{D0}H_{D4}r^4)} + \frac{H_{D4}(2F_{\sigma z}\omega + F_{\sigma z}^2 H_{F1}r^2 + H_{D0}H_{D4}r^4)\dot{r}^2}{2F_{\sigma z} r^2} \right]. \tag{4.7}$$

Setting $\dot{r}$ to zero in (4.7) we obtain the effective potential seen by the supertube moving in the BMPV background:

$$V(r) = (2\pi)^2 R_A \tau_{D2} \frac{F_{\sigma z} r^2 (F_{\sigma z}\mu_{F1} - H_{D4}r^2)^2}{2H_{D4}r^2(2F_{\sigma z}\omega + F_{\sigma z}^2 H_{F1}r^2 + H_{D0}H_{D4}r^4)} \geq 0\,. \tag{4.8}$$

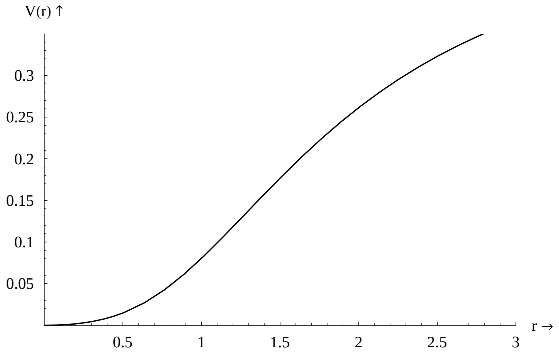

Figure 1: The potential (4.8) with parameters $(2\pi)^2 R_A \tau_{D2} = r_{D0} = r_{D4} = r_{F1} = 2\omega = 1$ and $F_{\sigma z} = \mu_{F1} = 0.1$. Here $\mu_{F1} F_{\sigma z} - r_{D4}^2 < 0$ and the potential is attractive for all r. The potential vanishes at $r = 0$.

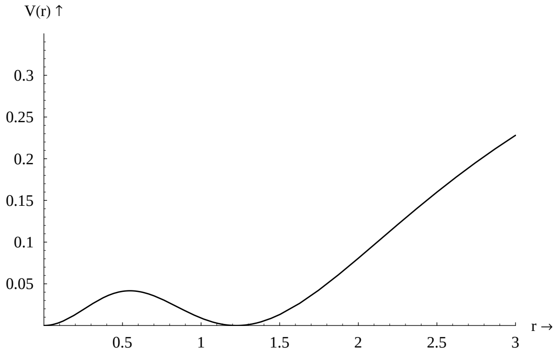

Figure 2: The potential with parameters $(2\pi)^2 R_A \tau_{D2} = r_{D0} = r_{F1} = 2\omega = 1, F_{\sigma z} = \mu_{F1} = 1, r_{D4} = 0.8$. Here $\mu_{F1} F_{\sigma z} - r_{D4}^2 > 0$ so that the potential has an additional zero at some $r > 0$.

Note that the denominator of (4.8) is strictly positive and does not vanish even at $r = 0$. Thus, V vanishes at $r = 0$ due to the explicit r^2 in the numerator. Since the potential is non-negative, we see that it is always attractive near $r = 0$.

Further from $r = 0$ the behavior is determined by the relative sizes of $F_{\sigma z}$, μ_{F1}, and r_{D4}^2. Note that $\lim_{r \to 0} H_{D4} r^2 = r_{D4}^2$. Thus, for $\mu_{F1} F_{\sigma z} > r_{D4}^2$ there is an additional zero at some $r > 0$. It is useful to express this relation in terms of the integer charges by noting that

$$\mu_{F1} F_{\sigma z} - r_{D4}^2 = \frac{g_A \alpha'^{3/2}}{R_A} (q_{D0} q_{F1} - N_{D4}) \,. \tag{4.9}$$

Thus, an additional zero appears for $q_{D0} q_{F1} > N_{D4}$. It is precisely at this additional zero that one finds the static supersymmetric configurations (with $\theta = \pi/2$) studied in [1]. Note that there are no supersymmetric configurations at $r = 0$, as the world-volume would be required to be null.

We see that such solutions are separated from $r = 0$ by a finite energy barrier whose height is controlled by $q_{D0} q_{F1} - N_{D4}$ and the various moduli. The height of this barrier is readily estimated by noting that the denominator of (4.8) has its minimum at the origin, and that this minimum is in turn greater than $2 r_{D4}^4 r_{D0}^2$. Thus we find:

$$\text{Barrier height} \quad < \tau_{D0} q_{D0} \frac{\ell^4}{(\alpha')^2} \frac{N_{D4}}{N_{D0}} \,. \tag{4.10}$$

Plots of the potential for typical cases with each sign of $q_{D0} q_{F1} - N_{D4}$ are shown in figures 1 and 2.

We may now ask under what circumstances a supertube will reach $r = 0$ and merge with the black hole. Let us recall from [1] that, due to the fact that the supertube angular momentum is $j = q_{D0} q_{F1}$, the sign of $q_{D0} q_{F1} - N_{D4}$ is of interest: adding tubes with $q_{D0} q_{F1} > 4 N_{D4}$ *can* result in an object with $(J + q_{D0} q_{F1})^2 > (N_{D0} + 2 q_{D0})(N_{F1} + 2 q_{F1}) N_{D4}$ (and thus violating the BMPV bound on angular momentum), while adding tubes with

$q_{D0}q_{F1} < 4N_{D4}$ will not violate this bound.[4] In the case $q_{D0}q_{F1} - N_{D4} < 0$ (so that we also have $q_{D0}q_{F1} - 4N_{D4} < 0$) a supertube with $\theta = \pi/2$ is attracted to the black hole, and we find in agreement with [1] that the objects readily merge.

Suppose on the other hand that $q_{D0}q_{F1} - N_{D4} > 0$ (as needed if we are to have $q_{D0}q_{F1} - 4N_{D4} > 0$), and that we begin with the supersymmetric configuration considered in [1]; i.e., one located at the $r > 0$ minimum of the potential V. To cause the supertube to merge with the black hole, we need only send it over the intervening hill. One might imagine doing so by gently raising the supertube over the barrier and then slowly lowering it down the other side so that it collides with the black hole having an energy ΔE above the BPS bound that is arbitrarily small in comparison to all other quantities. Such a process can lead to a final object which violates the BMPV angular momentum bound $J^2 < N_{D0}N_{D4}N_{F1}$. Furthermore, since the energy above extremality is arbitrarily small compared with all other quantities, it can also violate the similar bound which limits the angular momentum of the non-BPS Cvetic-Youm solutions [33, 34] described earlier in section 2. Similarly, an object in the minimum at $r > 0$ will eventually tunnel quantum mechanically to $r = 0$ and merge with the black hole.

The purist may wish to study the case where one simply gives the supertube a velocity toward $r = 0$ which is sufficiently great to carry it over the hill. Again, if the black hole begins with enough angular momentum, the final state has squared angular momentum $(J^f)^2 = (J + q_{D0}q_{F1})^2$ larger than the product of final charges: $N^f_{D0}N^f_{D4}N^f_{F1} = (N_{D0} + 2q_{D0})(N_{F1} + 2q_{F1})N_{D4}$. However, the barrier height now sets the minimum energy ΔE above the BPS bound for the final object.

Now, for $J_1 = J_2$ and near extremality, the Cvetic-Youm solutions satisfy

$$\begin{aligned} J_1^2 < \; & N_{D0}N_{D4}N_{F1} \times \\ & \times \left(1 + \Delta E \frac{2G_5}{\pi} \frac{r_{D4}^4 r_{F1}^4 + r_{D0}^4 r_{F1}^4 + r_{D4}^4 r_{D0}^4 + 2r_{D0}^2 r_{D4}^2 r_{F1}^2 (r_{F1}^2 + r_{D0}^2 + r_{D4}^2)}{r_{D0}^2 r_{D4}^2 r_{F1}^2 (r_{D4}^2 r_{F1}^2 + r_{D0}^2 r_{F1}^2 + r_{D4}^2 r_{D0}^2)} + \right. \\ & \left. \quad + \mathcal{O}(\Delta E)^2 \right). \end{aligned} \tag{4.11}$$

This somewhat complicated expression becomes simpler in the special case $N_{F1} = N_{D0} = N_{D4} = N$, where it may be written

$$J_1^2 < N^3 \left(1 + (\text{moduli}) \frac{\Delta E}{N} + \mathcal{O}(\Delta E)^2 \right), \tag{4.12}$$

where *moduli* denotes a function of the moduli alone; i.e., a factor independent of the charges q_{D0}, q_{F1}, N. Using (4.10) as an upper bound for ΔE, it is straightforward to arrange parameters so that the final object violates this bound as well. For example, for large N one may choose $q_{D0} > 2\sqrt{N}$ and $q_{F1} > (2 + \mathit{moduli})\sqrt{N}$. This is in contrast to

[4] In order that the final angular momenta will again be equal, we consider sending a pair of otherwise identical supertubes (each with charges q_{D0}, q_{F1}) toward the hole, with one tube lying in the ϕ_1 plane and the other lying in the ϕ_2 plane. This is the source of factors of 2 and 4 above that differ from those in [1]. Keeping the two angular momentum equal allows for a more direct comparison with the BMPV bound on angular momentum.

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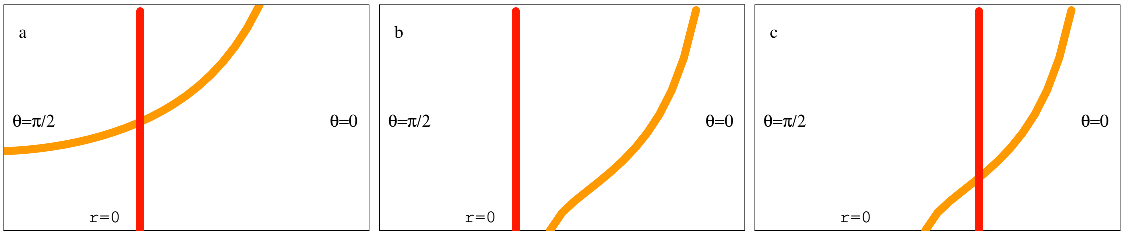

Figure 3: The curves along which the two terms in (4.15) vanish are drawn in the (r, θ) plane, with θ ranging over $[0, \pi/2]$ from right to left on each diagram and r increasing toward the top from zero at the bottom.. The straight vertical line shows the zeros of the second term; the location of this curve is controlled by j_2. The structure of the other curve is controlled by $q_{D0}q_{F1} - N_{D4}$. For $q_{D0}q_{F1} > N_{D4}$, the two curves always intersect as shown in figure *a*. For $q_{D0}q_{F1} < N_{D4}$, the curves may (figure *b*) or may not intersect (figure *c*) depending on the value of j_2.

the static analysis of [1] which indicated that the bound on the total angular momentum would not be violated.

In the case where we merely give the tube a finite kinetic energy, the detailed dynamics may be read off by expanding ΔE about $r = 0$. The leading terms are

$$\frac{\Delta E}{(2\pi)^2 R_A \tau_{D2}} = \frac{(F_{\sigma z}\mu_{F1} - r_{D4}^2)^2}{2A} r^2 + \frac{A}{2}\frac{\dot{r}^2}{r^4} + \text{higher order terms}\,, \tag{4.13}$$

where

$$A = \frac{r_{D4}^2(2F_{\sigma z}\omega + F_{\sigma z}^2 r_{F1}^2 + r_{D0}^2 r_{D4}^2)}{F_{\sigma z}} \qquad \text{is a constant}\,.$$

Due to the singular kinetic term, the supertube reaches $r = 0$ only at infinite t, just as one would expect for an object falling through the horizon of a black hole.

4.2 The general case: motion in θ

The special case $\theta = \pi/2$ is useful to build intuition, but has the disadvantage that (since motion is allowed only in r) the size of the supertube is forced to change if it moves. Since supertubes have a preferred size this tends to create a confining potential. Thus we may hope that by relaxing the restriction on θ we might find solutions in which the supertubes propagate freely from far away to the horizon of the black hole (without our needing to apply additional forces). Indeed, one might expect such a result from the fact that [1] found BPS configurations of fixed (small) charge arbitrarily close to the horizon. However, we will see that is in fact *not* the case due to magnetic interactions between the black hole and the supertube.

Proceeding along the same lines as above we find that the energy (see (4.2)) associated with the motion of the supertube is $E = \tau_{D0}|q_{D0}| + \tau_{F1}|q_{F1}|2\pi R_A + \Delta E$, where

$$\begin{aligned}\Delta E &= (2\pi)^2 R_A \tau_{D2} \times \\ &\times \left[\frac{H_{D4}(2F_{\sigma z}\omega\sin^2\theta + F_{\sigma z}^2 H_{F1} r^2 + H_{D0}H_{D4} r^4 \sin^2\theta)}{2F_{\sigma z} r^2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\dot{\phi}_2^{\,2}) + \right. \\ &\left. + \frac{F_{\sigma z}(F_{\sigma z}\mu_{F1} - H_{D4} r^2 \sin^2\theta)^2}{2H_{D4}\sin^2\theta(F_{\sigma z}^2 H_{F1} r^2 + 2\omega F_{\sigma z}\sin^2\theta + H_{D0}H_{D4} r^4 \sin^2\theta)}\right].\end{aligned} \tag{4.14}$$

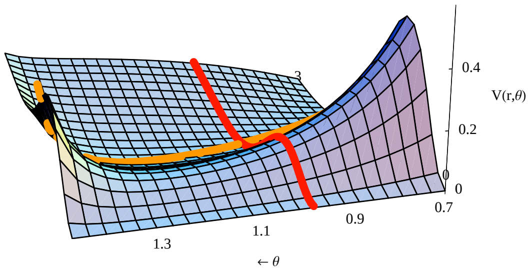

Figure 4: The potential as a function of r, θ with parameters set to $(2\pi)^2 R_A \tau_{D2} = r_{D0} = r_{D4} = r_{F1} = 2\omega = 1, F_{\sigma z} = 1, \mu_{F1} = 2.5$, and $j_2 = 0.7$. The coordinate r again ranges from 0 to 3 and θ ranges from 0.7 to $\frac{\pi}{2} - 0.1$. The heavy lines correspond to the curves shown figure 3a.

Here $\dot{\phi}_2$ can be replaced with j_2, the conserved angular momenta conjugate to ϕ_2 from (4.5). We obtain an effective potential by setting $\dot{r}, \dot{\theta}$ to zero in (4.14):

$$V(r,\theta) = \frac{(2\pi)^2 R_A \tau_{D2} F_{\sigma z}}{2H_{D4}(F_{\sigma z}^2 H_{F1} r^2 + 2\omega F_{\sigma z} \sin^2\theta + H_{D0} H_{D4} r^4 \sin^2\theta)} \times \\ \times \left[\frac{(F_{\sigma z}\mu_{F1} - H_{D4} r^2 \sin^2\theta)^2}{\sin^2\theta} + \frac{(j_2/(2\pi)^2 R_A \tau_{D2} - r_{D4}^2 \cos^2\theta)^2}{\cos^2\theta} \right]. \quad (4.15)$$

Note that the first term inside the square brackets will always vanish on some curve in the (r,θ) plane, while for small enough j_2 the second term will vanish at some value of θ (i.e., on another curve in the (r,θ) plane). The qualitative features of the dynamics are controlled by these two curves, for which three typical configurations are sketched in figure 3 below. For $q_{D0} q_{F1} > N_{D4}$ the curves will intersect as shown in figure 3a. At the intersection point the potential will vanish and have a local minimum. This minimum corresponds to the general static supersymmetric supertube considered in [1]. Note that the potential also vanishes at $r = 0$. Such a potential is plotted in Fig. 4 for typical values of the parameters.

In contrast, for $q_{D0} q_{F1} < N_{D4}$ the curves may (figure 3b) or may not intersect (figure 3c), depending on the value of j_2. In the case with no intersection, the potential vanishes only at $r = 0$.

5. Discussion: the final state

In the preceding sections we have studied the moduli-space scattering of a two-charge supertube moving in the type IIA background (with F1, D0, and D4-brane charges) dual to that of a rotating BPS D1-D5-P black hole [31, 32], also called a BMPV black hole after the authors of [32]. For simplicity, we restricted the analysis (as in [1]) to tubes preserving the ϕ_1 translation symmetry of the background (2.1) and to tubes for which the electric flux in the σ-direction vanishes. Symmetries and conversation laws guarantee that such an ansatz is consistent, and that one may substitute this ansatz into the action to obtain valid equations of motion. Such moduli space analyses take into account magnetic forces, which can play important roles in the dynamics.

Although we work near the BPS limit, we found that the motion of such tubes can be described by an effective potential V depending on two coordinates (r, θ). This effective potential is a result of the various conserved charges in the problem, and arises in a manner similar to the effective potential found by Peeters and Zamaklar [35] in their study of a field theoretic problem related by dualities to the scattering of two supertubes.

In our context, BPS supertube configurations arise at local minima of this potential (with $V = 0$). Though the potential also vanishes at the horizon ($r = 0$), magnetic forces cause these zeros to be separated by a potential barrier. Thus, any BPS supertube is stable to a sufficiently small perturbation and the perturbation does not result in merger of the tube with the black hole. However, since the potential barrier is of finite height, the supertube may be lifted over the barrier by additional forces (or may quantum mechanically tunnel through) so that the tube merges with the black hole having arbitrarily small energy above the BPS bound. This is in contrast to the conclusion of [1] (based on an analysis of exactly supersymmetric configurations) that only supertubes with $q_{D0}q_{F1} < N_{D4}$ could merge with the black hole. Here we remind the reader that q_{D0}, q_{F1} are integer charges carried by the supertube while N_{D4} is an integer charge carried by the black hole.

In the above, we have said that the tube and black hole "merge" if a moduli-space calculation yields a trajectory in which the tube reaches $r = 0$. Noting that inverse powers of r appear at various points in the calculation, the reader may legitimately ask if other effects, neglected above, might change this conclusion. However, an examination of equations (A.3, A.4) in Appendix A show that, within the test supertube approximation (in which back-reaction is ignored), there can be no such effects. The point is that the only terms in the action which diverge as $r \to 0$, are those involving $\dot{r}$ via (A.4), and such terms always involve $\dot{r}/r$. Thus, the apparent divergences can be removed by transforming to the non-singular modui-space coordinate $\rho = \ln r$. It is then clear that the action may be expanded in powers of $\dot{\rho}$ and the other velocities, with the coefficients of each term being bounded near $\rho = -\infty$ (i.e., $r = 0$). Thus, we may consistently neglect effects associated with higher powers of the velocity. As a result, we see that in the test-tube approximation we can indeed arrange our tubes to cross the future horizon of, and thus merge with, the BMPV black hole.

Although back-reaction may well be important in understanding the dynamics of the merger, the fincal result is of course constrained by conservation laws. We have considered supertubes with $j_2 = 0$, but $j_1 \neq 0$ while the black hole background has $J_1 = J_2$. Since for motion on moduli space there is no radiation at leading order in velocities [50, 51], the angular momenta of the object resulting from the merger are clearly not equal. Thus, the final state cannot be just another BMPV black hole. In addition, because supertubes with $q_{D0}q_{F1} > 4N_{D4}$ can merge with the black hole while having arbitrarily small energy above the BPS bound, we find in contrast to [1] that, depending on both the supertube and the original black hole, the object resulting from the merger may also violate the upper bound (2.11) on the magnitude of the angular momentum associated with BMPV black holes.

Now, due to the kinetic energy inherent in our scattering analysis, the final state will not saturate the BPS bound. Thus, we should consider the non-extreme generalizations of BMPV (i.e., the Cvetic-Youm solutions [33, 34] described in section 2) as potential final states. Note, however, that for any angular momentum the departure of our supertube from the BPS bound may be arbitrarily small in comparison with its angular momentum (which does not vanish in the BPS limit). As a result, we can also arrange to violate the constraints on both $J_1 - J_2$ and on $J_1 + J_2$ associated with non-extreme black holes. Thus, a Cvetic-Youm solution cannot by itself form the final state of our collision.

Now, it is plausible that the Cvetic-Youm black holes are the only black hole solutions that include small deformations of BMPV.[5] Under such an assumption, we would be forced to conclude that sending an arbitrarily small supertube into our black hole does not result in a small change. Such a situation might arise if our supertube triggers an instability of the above family of black hole solutions. While one would expect BPS solutions like BMPV to be at least marginally stable at linear order, the non-extreme solutions of [33, 34] could well be subject to linear instabilities. Such an instability arbitrarily close to the BPS limit is reminiscent of (and might perhaps be connected to) the gyrational instability of the 5+1 dimensional BMPV black string suggested in [53].

The activation of an instability would take us out of the regime in which the scattering is described by motion on moduli space and could lead to a variety of effects. For example, the activation of an instability might lead to radiation of conserved quantities to infinity. However, noting that the massless radiation carries no charge, we see that the energy carried to infinity will be small since the mass of the remaining object must still satisfy the BPS bound. Furthermore, radiation can carry angular momentum only if it breaks the corresponding rotational symmetry. Since our supertubes preserve rotational symmetry in ϕ_1, the radiation can change only the angular momentum in the ϕ_2 plane. For $j_1 > 0$, such radiation might restore self-duality of the black hole's angular momentum only by leading to an increase in the total angular momentum. But if the final state were to be a BMPV black hole, such an increase of angular momentum would decrease the horizon area. It is therefore forbidden by the area theorem and the second law of thermodynamics. Thus, at least for $j_1 > 0$, radiation alone does not hold the key to the final state.

Another possibility is that the instability may lead to a non-perturbative change in the black hole. For example, the instability might in principle cause the spherical (S^3) horizon of the BMPV hole to become $S^1 \times S^2$, leading to a black ring. This is analogous to one of the scenarios discussed by Emparan and Myers in [9] with regard to their instability of ultraspinning black holes in $d \geq 6$ spacetime dimensions.

Now, a final state with a black ring alone is again forbidden by the area theorem. The point here is that, as noted in section 2, there is an "entropy gap" between the entropy of a BMPV black hole and the entropies of nearby black rings. Thus, for any

[5]Though it is also plausible that there are other non-extreme black holes with less symmetry as was conjectured in [52]. If stable, such black holes would provide additional candidates for the final state resulting from our collision.

BMPV black hole, there is an open set in the space of conserved charges such that every black ring in this open set has smaller entropy than does the original black hole. As a result, merging a sufficiently small supertube with the black hole cannot result in a black ring.

However, one should again allow for the effects of radiation emitted during the supposed transition. Let us first consider the case where the original black hole has large angular momentum, so that its area is nearly zero. Then since $N_{D1}N_{D5}N_p - \mathcal{N}_{D1}\mathcal{N}_5\mathcal{N}_p$ is quadratic in the dipole charges for small dipoles whereas the coefficient of J_1 is cubic, it is clear from (2.20) that by radiating a sufficient amount of angular momentum J_2 we may arrive at a black ring of larger entropy (and with small dipole charges). Thus, if the original black hole angular momentum is large (as in cases leading to a violation of the BMPV angular momentum bound (2.16)), the merger may result in a black ring together with some radiation at infinity.

On the other hand, if the angular momentum of the original black hole is small, its entropy will be greater than the maximum entropy (2.22) of a black ring with the same conserved charges. Thus, for this case the area theorem again forbids the final state from being a single black ring with radiation at infinity.

Nonetheless, we note that a *pair* of black rings of the sort described by Gauntlett and Gutowski [40] could provide the required final state. As shown in [40], one can find pairs whose charges are equal to those of a BMPV black hole but for which the total horizon area of the pair exceeds that of the original BMPV hole by a finite amount. Thus, even with the small change in the charges and angular momentum associated with the absorption of the supertube, a two black ring final state is entropically allowed.[6] Thus, the merger may well induce a fragmentation of the original black hole into multiple black objects.[7]

This fragmentation scenario is also similar to one of the options outlined by Emparan and Myers [9] in discussing their instability of ultraspinning black holes in $d \geq 6$ spacetime dimensions. However, here we see that this is the *only* available scenario based on known solutions. Thus, if our scenario is confirmed, it will lead to a picture of black hole interactions in $d > 4$ spacetime dimensions which is even more similar to that associated with particle scattering than was previously expected. In addition to the possibility that collisions result in the fusion of two particles into one of greater mass, one must also allow the possibility of black hole fission. It would be interesting to examine this possibility either via perturbative study of excitations about a Cvetic-Youm black hole, or through numerical simulations.

[6] Various degenerate cases are also allowed. For example, one cannot rule out a final end state similar to the initial state, containing a BMPV black hole with the original charges and a supertube (which may be viewed as a degenerate limit of a black ring [39]). However, such a scenario would yield zero net increase in entropy — a result which seems rather unlikely. Cases may also arise where a non-degenerate black ring together with a BMPV black hole is allowed.

[7] The pedant may wonder what will happen if the two black rings then collide and merge. However, we note that any such collision requires a breaking of the ϕ_1 rotation symmetry. Thus, it cannot occur in the scenario envisioned above. In another setting without such a symmetry, radiation could carry away both J_1 and J_2 angular momenta, leaving behind a slowly-rotating (or non-rotating) black hole.

Acknowledgments

We would like to thank Henriette Elvang, Per Kraus, Anshuman Maharana, David Mateos, and Harvey Reall for many valuable discussions. We also thank Iosif Bena for useful correspondence. This work was supported in part by NSF grant PHY0354978, by funds from the University of California, and by the Perimeter Institute for Theoretical Physics.

A. The induced fields and the Dirac-Born-Infeld action

In this appendix we calculate the Dirac-Born-Infeld part of the effective action for a supertube propagating in the type IIA dual (3.5) of the BMPV background [31, 32]. We align the D2 brane such that two of the D2-brane coordinates t, z coincide with the spacetime coordinates,

$$\begin{aligned} t_{\text{spacetime}} &= t_{\text{supertube}} \\ z_{\text{spacetime}} &= z_{\text{supertube}} \,, \end{aligned} \tag{A.1}$$

and restrict our analysis to circular cross sections so that $r, \theta, \phi_1, \phi_2$ are functions of t alone.

As a first step, we calculate the fields induced on the supertube by the embedding in our background. We denote induced fields by lower case letters and background fields by upper case letters, e.g.,

$$t_{a\dots b} = \partial_a X^\mu \dots \partial_b X^\nu T_{\mu\dots\nu} \,, \tag{A.2}$$

for a background field $T_{\mu\dots\nu}$. In the equations below, lower case latin indices i, j run over r, θ, ϕ_1 and ϕ_2. We find:

$$\begin{aligned}
g_{00} &= G_{00} + 2G_{i0}\frac{\partial x^i}{\partial t} + G_{ij}\frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial t} = -H_{D0}^{-1/2}H_{D4}^{-1/2}H_{F1}^{-1}(1+\gamma_t)^2 + (H_{D0}H_{D4})^{1/2}\Delta_{tt} \\
g_{0\sigma} &= G_{0i}\frac{\partial x^i}{\partial \sigma} + G_{ij}\frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial \sigma} = -H_{D0}^{-1/2}H_{D4}^{-1/2}H_{F1}^{-1}\gamma_\sigma(1+\gamma_t) + (H_{D0}H_{D4})^{1/2}\Delta_{\sigma t} \\
g_{0z} &= G_{0z} + G_{iz}\frac{\partial x^i}{\partial t} = 0 \\
g_{z\sigma} &= G_{zi}\frac{\partial x^i}{\partial \sigma} = 0 \\
g_{zz} &= G_{zz} = (H_{D0}H_{D4})^{1/2}H_{F1}^{-1} \\
g_{\sigma\sigma} &= G_{ij}\frac{\partial x^i}{\partial \sigma}\frac{\partial x^j}{\partial \sigma} = (H_{D0}H_{D4})^{1/2}\Delta_{\sigma\sigma} - H_{F1}^{-1}H_{D0}^{-1/2}H_{D4}^{-1/2}\gamma_\sigma^2 \\
b_{t\sigma} &= B_{ti}\frac{\partial x^i}{\partial \sigma} + B_{ij}\frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial \sigma} = 0 \\
b_{tz} &= B_{tz} + B_{iz}\frac{\partial x^i}{\partial t} = (H_{F1}^{-1} - 1) + \gamma_t H_{F1}^{-1} \\
b_{\sigma z} &= B_{iz}\frac{\partial x^i}{\partial \sigma} = \gamma_\sigma H_{F1}^{-1} \\
c_{1t} &= C_{1t} + C_{1i}\frac{\partial x^i}{\partial t} = (H_{D0}^{-1} - 1) + \gamma_t H_{D0}^{-1}
\end{aligned}$$

$$c_{1\sigma} = C_{1i}\frac{\partial x^i}{\partial\sigma} = \gamma_\sigma H_{D0}^{-1} \tag{A.3}$$
$$c_{3t\sigma z} = C_{3tiz}\frac{\partial x^i}{\partial\sigma} + C_{3ijz}\frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial\sigma} = \gamma_\sigma H_{F1}^{-1} - (H_{D4}-1)r^2\cos^2\theta\left(\frac{\partial\phi_1}{\partial t}\frac{\partial\phi_2}{\partial\sigma} - \frac{\partial\phi_1}{\partial\sigma}\frac{\partial\phi_2}{\partial t}\right),$$

where γ_ξ and $\Delta_{\xi\eta}$ is defined in (3.14) and (3.16) respectively. Note that since we required the velocities on the torus (T^4) to vanish, we have

$$G_{ij}\frac{\partial x^i}{\partial\xi}\frac{\partial x^j}{\partial\eta} = (H_{D0}H_{D4})^{1/2}\Delta_{\xi\eta} - H_{F1}^{-1}H_{D0}^{-1/2}H_{D4}^{-1/2}\gamma_\xi\gamma_\eta\,. \tag{A.4}$$

To evaluate the Dirac-Born-Infeld part of the action, we must compute the determinant $\sqrt{-\det(g+b+F)}$. We wish to perform an expansion in the velocities and the fluctuations $\delta F_{tz} := (F_{tz}-1)$ and $F_{t\sigma}$ to second order. At zeroth order the contribution is

$$H_{D0}^{-1/2}H_{D4}^{-1/2}H_{F1}^{-1}F_{\sigma z}^2\,, \tag{A.5}$$

while at linear order contributing terms are

$$\frac{2}{H_{D4}^{1/2}H_{D0}^{1/2}H_{F1}}\left[F_{\sigma z}^2\gamma_t + F_{\sigma z}H_{D0}H_{D4}\Delta_{\sigma t} - \{F_{\sigma z}\gamma_\sigma + H_{D0}H_{D4}\Delta_{\sigma\sigma}\}\,\delta F_{tz}\right], \tag{A.6}$$

and at second order we find

$$\begin{aligned}
&- H_{D4}^{1/2}H_{D0}^{1/2}\Delta_{tt}\left(F_{\sigma z}^2 + 2F_{\sigma z}H_{F1}^{-1}\gamma_\sigma + H_{D4}H_{D0}H_{F1}^{-1}\Delta_{\sigma\sigma}\right) + \\
&+ H_{D4}^{-1/2}H_{D0}^{-1/2}H_{F1}^{-1}\left[(F_{\sigma z}\gamma_t + H_{D0}H_{D4}\Delta_{\sigma t})^2 + \left(\gamma_\sigma^2 - H_{D4}H_{D0}H_{F1}\Delta_{\sigma\sigma}\right)\delta F_{tz}^2\right] + \\
&+ 2H_{D0}^{1/2}H_{D4}^{1/2}H_{F1}^{-1}\left\{\Delta_{\sigma t}\left(\gamma_\sigma + H_{F1}F_{\sigma z}\right) - \frac{\gamma_\sigma\gamma_t F_{\sigma z}}{H_{D0}H_{D4}} - \Delta_{\sigma\sigma}\gamma_t\right\}\delta F_{tz} - \\
&- H_{D0}^{1/2}H_{D4}^{1/2}H_{F1}^{-1}F_{t\sigma}^2\,.
\end{aligned} \tag{A.7}$$

Substituting (A.5–A.7) into the Dirac-Born-Infeld action we find

$$\begin{aligned}
S_{DBI} &= \\
&= \tau_{D2}\int dtdzd\sigma\Bigg[-H_{D0}^{-1}F_{\sigma z} - H_{D4}\Delta_{\sigma t} + \left\{\frac{H_{D4}\Delta_{\sigma\sigma}}{F_{\sigma z}} + \gamma_\sigma H_{D0}^{-1}\right\}\delta F_{tz} - H_{D0}^{-1}F_{\sigma z}\gamma_t + \\
&\qquad + \frac{H_{D4}H_{F1}\Sigma}{2F_{\sigma z}}\left(\Delta_{tt} - \frac{2\Delta_{\sigma t}\delta F_{tz}}{F_{\sigma z}} + \frac{\Delta_{\sigma\sigma}\delta F_{tz}^2}{F_{\sigma z}^2}\right) + \frac{H_{D4}F_{t\sigma}^2}{2F_{\sigma z}}\Bigg].
\end{aligned} \tag{A.8}$$

Adding the Wess-Zumino terms (3.13) to S_{DBI} we obtain the full action (3.15):

$$\begin{aligned}
S = &\tau_{D2}\int dtdzd\sigma\Bigg[-F_{\sigma z} - H_{D4}\Delta_{\sigma t} + \left\{\frac{H_{D4}\Delta_{\sigma\sigma}}{F_{\sigma z}}\right\}\delta F_{tz} + \frac{H_{D4}F_{t\sigma}^2}{2F_{\sigma z}} - (H_{D4}-1)r^2\cos^2\theta\times \\
&\times\left(\frac{\partial\phi_1}{\partial t}\frac{\partial\phi_2}{\partial\sigma} - \frac{\partial\phi_1}{\partial\sigma}\frac{\partial\phi_2}{\partial t}\right) + \frac{H_{D4}H_{F1}\Sigma}{2F_{\sigma z}}\left(\Delta_{tt} - \frac{2\Delta_{\sigma t}\delta F_{tz}}{F_{\sigma z}} + \frac{\Delta_{\sigma\sigma}\delta F_{tz}^2}{F_{\sigma z}^2}\right)\Bigg].
\end{aligned} \tag{A.9}$$

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